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On Approximate Solutions for Unsteady Conduction in Slabs with Uniform Heat Flux

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In this paper, the Transversal Method of Lines (TMOL) or Rothe's method is employed to obtain analytical expressions of simple form for the unsteady one-dimensional heat conduction in a slab. Initially, the slab is maintained at a uniform temperature, and then a uniform heat flux is applied to its surfaces. Implementation of TMOL generates a sequence of adjoint ordinary differential equations, where the spatial coordinate is the only independent variable and the time becomes a parameter. In spite of the anticipated expectations that the semi-discrete solutions produced by TMOL would yield accurate temperature responses for short times only, detailed calculations demonstrate the opposite trend. Surprisingly, the temperature results associated with two equal time steps are excellent not only for short times, but during the entire heating period.

1. INTRODUCTION

The method of separation of variables, the Green's function procedure, and the integral transform technique are the most commonly used analytical schemes for the analysis of unsteady heat conduction in regular bodies [1–3]. The solution produced by any of these three methods consists on convergent infinite Fourier-type series. In general, the use of the method of separation of variables is not convenient whenever a parabolic partial differential equation (PDE) and/or the boundary conditions involve non-homogeneities. In contrast, both the Green's function procedure and the integral transform technique operate satisfactorily and provide alternative approaches for the analytical solutions of these non-homogeneous problems.

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One important feature of the solutions by Fourier series is that they converge very slowly for small values of time, implying the necessity of retaining a large number of terms in the series to achieve good accuracy in the local temperature calculations. Because of this restriction, Fourier series solutions are not convenient for numerical computations involving short times [2].

Usually the above cited obstacle is overcome by using the Laplace transform to redefine the infinite series in terms of some integral functions that converge fast for short times, specially those functions commonly present in heat conduction solutions involving semi-infinite domains, e.g., the error function. Sometimes, the series route is discarded and the semi-infinite body solution by itself is employed to estimate the early transient behavior. However, in most cases the resulting expressions are difficult to evaluate and tables and/or graphics are needed to estimate the desired temperature values. Moreover, the integrals and derivatives of these expressions, which are frequently needed in analyses, are cumbersome to manipulate. Another important aspect that needs to be addressed is the range of validity of these type of approximate solutions which seemingly has not been studied in detail yet.

For certain problems in applied physics and engineering that require only late transient behavior, the resulting infinite series solution is frequently truncated and reduced to its minimum expression, giving the so-called one-term solution or long-time solution, so popular in the heat conduction literature because of its simplicity [2, 3]. Understandably, there is no homologous simple one-term solution for short times.

With this background in mind, the central objective of this paper is to propose an alternative semi-analytic technique to supplement the long-time solution with simple formulas that permits fast and reliable calculations of the temperature response of regular bodies for short times. Parallel to this, the domain of validity of the proposed approach as well as other widely used short-time approximate solutions are also studied exhaustively for the case of a slab.

The Transversal Method of Lines (TMOL) or Rothe's method [4] seems to be an adequate technique to achieve our goal. In this paper, TMOL is applied to the one-dimensional, initial boundary value problem (IBVP) of heat conduction in a slab subjected to iso-heat flux at the surfaces, i.e., Neumann boundary condition. By virtue of this method the time-derivative in the parabolic PDE is discretized with a standard, first-order time-accurate, finite-difference formulation. The original PDE is therefore replaced by an adjoint ordinary differential equation (ODE) together with the imposed boundary conditions, where the time acts as a given parameter. As a result of this, the temporal-spatial computational domain is converted into a reduced spatial computational domain. Then, under these premises the integration of the resulting two-point boundary value problem (2-BVP) at each time level may be carried out analytically without difficulty.

Our article is organized as follows. In Section 2, we present the mathematical formulation of the problem. An outline of how the method of separation of variables is applied to obtain the exact solution of the problem is shown in Section 3. This exact solution is used later as a baseline solution for comparison purposes. Some popular approximate techniques able to compute the transient behavior of the problem are considered in Section 4. In Section 5, we present the proposed methodology and its variations. Here TMOL is implemented to derive simple solutions which are valid, in principle, for short times. To show the applicability of the proposed formulas, in Section 6, we report some results in tabular and graphical form. Finally, in Section 7 we give our concluding remarks.

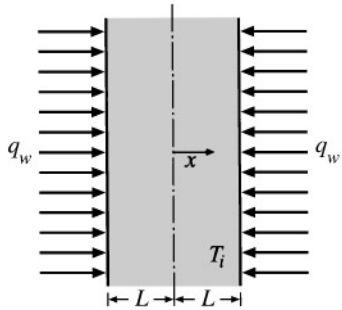

FIG. 1. Schematic view of the physical system.

2. MATHEMATICAL FORMULATION

A slab of thickness $2L$ possesses a uniform temperature T_i initially, $t \leq 0$. At $t = 0$, an uniform heat flux, q_w, is suddenly imposed to both surfaces of the slab, i.e., $x = \pm L$ (see Fig. 1). The time-dependent, one-dimensional heat conduction equation, along with the initial and Neumann boundary conditions, is expressed as

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial X^2} \qquad \text{in } 0 \leq X \leq 1, \tau > 0, \tag{1}$$

$$\phi = 0 \qquad \text{for } \tau = 0, 0 \leq X \leq 1, \tag{2}$$

$$\frac{\partial \phi}{\partial X} = 0 \qquad \text{at } X = 0, \tau > 0, \tag{3}$$

$$\frac{\partial \phi}{\partial X} = 1 \qquad \text{at } X = 1, \tau > 0. \tag{4}$$

The assumption of materials whose thermo-physical properties are not affected by temperature leads to the adoption of the dimensionless variables

$$\phi = \frac{T - T_i}{T_c}, \qquad X = \frac{x}{L}, \qquad \text{and} \qquad \tau = \frac{\alpha t}{L^2}, \tag{5}$$

where the characteristic temperature is denoted by $T_c = q_w L/k$.

3. EXACT SOLUTION

In this section, we briefly discuss the method of separation of variables which is commonly used to obtain the exact solution of the unsteady heat conduction problem. This exact solution is later employed for comparison purposes.

3.1. *Method of Separation of Variables*

The dimensionless temperature distribution, $\phi(X, \tau)$, via the method of separation of variables, will be considered as the baseline solution. The superposition of solutions given by the relations

$$\phi(X, \tau) = f_1(X, \tau) + f_2(X) + f_3(\tau), \tag{6}$$

can eliminate the difficulty arising from the non-homogeneous boundary condition, Eq. (4). Thus, the analytical full-series (FS) solution is taken directly from the textbook by Arpaci [2]

$$\phi(X,\tau) = \tau - \frac{1}{2}\left(\frac{1}{3} - X^2\right) - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{\mu_n^2}\cos(\mu_n X)e^{-\mu_n^2\tau}, \tag{7}$$

where $\mu_n = n\pi, n = 1, 2, 3, \ldots, \infty$, are the eigenvalues.

An important quantity related to the calculation of the total heat transfer rate and the bulk thermal energy is the dimensionless mean temperature

$$\bar{\phi}(\tau) = \int_0^1 \phi(\xi,\tau)\, d\xi = \tau, \tag{8}$$

which presents its usual unitary slope.

4. TRADITIONAL APPROXIMATE TECHNIQUES

In this section, we examine the one-term-of-series (OTS) solution, the Laplace transform short-time (LTST) solution, and the semi-infinite body (SIB) solution which are also employed later for comparison purposes. The first approach is very popular among heat transfer analysts because it permits us to obtain long-time solutions easily. In contrast, the other two approaches are the most frequently recommended in the specialized literature to get short-time solutions.

4.1. *One-Term-of-Series Solution*

The pressing characteristic of Fourier series, like Eq. (7), for purposes of numerical evaluation of local temperatures, is that they tend to converge rapidly for very long times allowing the retention of only one term, e.g., OTS solution,

$$\Phi(X,\tau) = \tau - \frac{1}{2}\left(\frac{1}{3} - X^2\right) + \frac{2}{\pi^2}\cos(\pi X)e^{-\pi^2\tau}, \tag{9}$$

where, despite the truncation, the corresponding mean temperature $\bar{\Phi}(\tau)$ still yields the correct magnitude

$$\bar{\Phi}(\tau) = \int_0^1 \Phi(\xi,\tau)\, d\xi = \tau, \tag{10}$$

regardless of the value of τ. This expression is indicative of the thermal energy of the system.

In passing, we should mention that, contrarily to their nice long-time behavior, Fourier series show severe divergence patterns for short and even moderate times when the series is truncated to few terms. This situation is so abnormal that the numerical evaluation of the one-term truncated series, i.e., Eq. (9), does not meet the initial condition. In fact, for very short times the evaluated local temperatures deviate from the initial condition drastically.

4.2. *Laplace Transform Solution*

The Laplace transform is frequently employed to obtain solutions for small levels of time. Regardless of the more advanced mathematical knowledge required, the basic concept behind this type of short-time solution technique is very simple. In fact, it amounts to an appropriate series expansion of the transformed solution in terms of the Laplace parameter, s. Based on the definition of Laplace transforms, it can be argued that the product $s\tau$ must be finite [2]. This implies

$$\text{short } \tau \rightarrow \text{large } s,$$
$$\text{large } \tau \rightarrow \text{short } s.$$

Focusing on short times, i.e., large values of s, the resulting anti-transformed series solution is usually expressed in terms of combinations of the complementary error function and its integrals. For the slab, the first term of the LTST series can be taken from the textbook by Luikov [3],

$$\Phi(X,\tau) = 2\sqrt{\tau}\left(I\,\mathrm{erfc}\left(\frac{1-X}{2\sqrt{\tau}}\right) + I\,\mathrm{erfc}\left(\frac{1+X}{2\sqrt{\tau}}\right)\right), \tag{11}$$

where $\mathrm{erfc}(z)$ is the complementary error function, given by

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-\xi^2}\,d\xi,$$

and $I\,\mathrm{erfc}(z) = \int_z^\infty \mathrm{erfc}(\xi)\,d\xi$, is the first integral of $\mathrm{erfc}(z)$. The latter can be computed by the recurrence relation

$$I\mathrm{erfc}(z) = \frac{e^{-z^2}}{\sqrt{\pi}} - z\,\mathrm{erfc}(z).$$

After some calculations, the corresponding mean temperature is found to be

$$\bar{\Phi} = \tau + 2\sqrt{\frac{\tau}{\pi}}e^{-1/\tau} - (2+\tau)\,\mathrm{erfc}\left(\frac{1}{\sqrt{\tau}}\right), \tag{12}$$

where the last two terms can be interpreted as a measure of the error in the prediction of the bulk thermal energy of the system as time progresses. Observe that as $\tau \rightarrow 0$, the LTST solution satisfies the exact result, i.e., $\bar{\Phi}(\tau) \rightarrow \bar{\phi}(\tau) = \tau$.

4.3. *Semi-Infinite Body Solution*

Because of its inherent simplicity, the SIB solution is also often employed to compute the evolution of temperature for small values of time. Essentially, its applicability relies on the fact that the thermal behavior of simple bodies resemble the thermal response of a semi-infinite body as $\tau \rightarrow 0$. However, it is expected that the range of validity of the SIB formulation for finite bodies should be restricted to very short times, when the effect of the other boundary condition has not been felt yet. The formula presented in Luikov [3], involving the first term of the LTST solution, takes the form

$$\Phi(X, \tau) = 2\sqrt{\tau}\, I \operatorname{erfc}\left(\frac{1 - X}{2\sqrt{\tau}}\right). \tag{13}$$

Similarly, following the same procedure delineated for the LTST formulation, the corresponding mean temperature can be expressed by

$$\bar{\Phi}(\tau) = \tau + \sqrt{\frac{\tau}{\pi}} e^{-1/4\tau} - \left(\frac{1}{2} + \tau\right) \operatorname{erfc}\left(\frac{1}{2\sqrt{\tau}}\right), \tag{14}$$

where again the last two terms account for a deflection from the correct global thermal energy of the system, i.e., $\bar{\Phi}(\tau) = \tau$, as τ increases.

5. TRANSVERSAL METHOD OF LINES

In this section, we propose a hybrid technique known as the Transversal Method of Lines (TMOL) or Rothe's method [4] to estimate the transient temperature response of simple bodies subjected to uniform heat flux at the surface. It will be shown that, depending of the admissible error in the TMOL-type approximate solutions, the formal short-time range of validity of the semi-analytical formulas derived in this section can be safely extended to the entire heating period.

The Transversal Method of Lines (TMOL) [4] will be applied now to the parabolic PDE, Eq. (1). Accordingly, the temporal derivative will be replaced by a two-point backward finite difference formula while leaving the space derivative in its continuous form. This procedure converts Eq. (1) into the following differential-difference equation

$$\frac{\Phi(X, \tau) - \Phi(X, \tau - \Delta\tau)}{\Delta\tau} = \frac{\partial^2 \Phi}{\partial X^2}(X, \tau), \tag{15}$$

which is a first-order time-accurate consistent representation of Eq. (1). Conceptually, it may be realized that the consistency of the above semi-discrete equation guarantees its equivalence with Eq. (1) in the limit $\Delta\tau \to 0$ [5]. Here, the time-truncation error is given by

$$\frac{\Delta\tau}{2} \frac{\partial^2 \Phi}{\partial \tau^2}(X, \sigma), \qquad \text{with } \sigma \in (\tau - \Delta\tau, \tau).$$

A more important question that needs to be answered belongs to the convergence of the solution of the approximate Eq. (15) to the exact solution of the original PDE, Eq. (1). This statement may be rephrased as follows: Under which conditions the solution of Eq. (15) approximates the exact solution of Eq. (1) [6], namely Eq. (6), as $\Delta\tau \to 0$? Some mathematical aspects of the method have been already extensively studied, see, for example, [7]. Nevertheless, we present some practical results regarding the convergence of the error associated with the solution in Section 6.

Next, consider the crudest attainable TMOL representation, i.e., one-step TMOL (1-TMOL). Literally, this simplest TMOL scheme jumps from the initial condition ($\tau = 0$) to the specified time τ when the solution is needed. Then, recognizing that for this stage the time step $\Delta\tau = \tau$ and introducing the initial condition, i.e., Eq. (2), into Eq. (15), leads to the following adjoint, second-order homogeneous ODE

$$\frac{d^2\Phi}{dX^2} - m_1^2\Phi = 0. \tag{16}$$

Here, $m_1^2 = 1/\tau > 0$, acts as some sort of thermo-geometric parameter which varies inversely proportional with the dimensionless time step, τ. From a physical standpoint, attention must be paid to the fact that $m_1^2 = L^2/\alpha t$ accounts for the thermal diffusivity of the material, α, the semi-thickness of the slab, L, and more importantly the actual time of calculation, t.

Moreover, Eq. (16) has to be supplemented with the same boundary conditions associated with the original PDE, i.e., Eqs. (3)–(4), rewritten as

$$\frac{d\Phi}{dX} = 0 \qquad \text{at } X = 0, \tag{17}$$

$$\frac{d\Phi}{dX} = 1 \qquad \text{at } X = 1. \tag{18}$$

In view of the foregoing, 1-TMOL has transformed the original IBVP into a simpler two-point boundary value problem (2-BVP) whose solution can be easily obtained analytically.

From a different optic, it is interesting to underline that the mathematical reformulation given by Eq. (16), together with the boundary conditions, Eqs. (17)–(18), is analogous to the mathematical description of the conductive-convective heat transfer from a straight fin of uniform cross section to a surrounding fluid. To comply with the analogy, this reference fin must have constant thermophysical properties, imposed heat flux at the base, and no heat loss through its tip.

Next, the analytical solution of the 1-TMOL formulation, Eqs. (16)–(18), may be written immediately as

$$\Phi(X, \tau) = C(m_1)\cosh(m_1 X), \tag{19}$$

where $C(m_1)$ takes the form

$$C(m_1) = \frac{1}{m_1 \sinh(m_1)},$$

and obviously the time dependence of the solution is manifested through the parameter $m_1 = m_1(\tau)$.

Another equally simple TMOL scheme that could be explored now is the one that reaches the solution at τ, implementing two steps of equal size, i.e., $\tau/2$. Utilization of the same procedure as above results in the set of equations

$$\frac{\Phi(X, \tau/2) - \Phi(X, 0)}{\tau/2} = \frac{\partial^2\Phi}{\partial X^2}(X, \tau/2),$$

$$\frac{\Phi(X, \tau) - \Phi(X, \tau/2)}{\tau/2} = \frac{\partial^2\Phi}{\partial X^2}(X, \tau),$$

which has to satisfy the same boundary conditions Eqs. (17)–(18). Next, inserting the initial condition Eq. (2), the first equation reduces to the homogeneous ODE,

$$\frac{d^2\Phi}{dX^2} - m_2^2\Phi = 0,$$

being $m_2^2 = 2/\tau > 0$. Accordingly, Eq. (19) satisfies the above ODE subjected to Eqs. (17)–(18), but in terms of m_2 instead. Introducing this information into the second equation, we generate the following linear non-homogeneous ODE,

$$\frac{d^2\Phi}{dX^2} - m_2^2\Phi = -m_2^2 C(m_2)\cosh(m_2 X). \tag{20}$$

Within the framework of the two-step TMOL (2-TMOL), $\Phi(X, \tau)$ is represented by the analytical solution of Eq. (20) subjected to the boundary conditions Eqs. (17)–(18), yielding

$$\Phi(X, \tau) = A(m_2)\cosh(m_2 X) - B(m_2)\, m_2 X \sinh(m_2 X), \tag{21}$$

where for conciseness $A(m_2)$ and $B(m_2)$ have been defined as

$$A(m_2) = B(m_2)(3 + m_2 \operatorname{cotanh}(m_2)) \qquad \text{and} \qquad B(m_2) = \frac{C(m_2)}{2},$$

respectively. From the perspective of numerical analysis, it is important to realize that Eq. (21) still constitutes a first-order time-accurate representation of $\phi(X, \tau)$.

At this point, a different avenue for improvement will be pursued. Because Eqs. (19) and (21) tacitly set forth semi-discrete solutions of $\phi(X, \tau)$ at two different time-steps, namely, $\tau/2$ and τ, we can improve the formal accuracy of the procedure by performing a Richardson Extrapolation (RE) [5]. Skipping the derivation for clarity, the end result is written as

$$\Phi(X, \tau) = 2A(m_2)\cosh(m_2 X) - 2B(m_2)\, m_2 X \sinh(m_2 X) - C(m_1)\cosh(m_1 X). \tag{22}$$

Finally, the expression for the mean temperature distribution may be computed from any of the pertinent solutions, Eqs. (19), (21), or (22). Lemma A.1 (see Appendix A), ratifies that all of these equations lead to the same exact answer, i.e.,

$$\bar{\Phi}(\tau) = \int_0^1 \Phi(\xi, \tau)\, d\xi = \tau, \tag{23}$$

which indicates invariance with respect to the level of approximation selected. An explanation of this unexpected or surprising accurate prediction of the mean temperature is given in the lemma: along with its striking behavior, the mean value of the time-truncation error is zero. Owing to the peculiar characteristics of TMOL, the integral of the deviations formed between the approximate and the exact temperature distributions cancel out, yielding the correct answer.

6. DISCUSSION OF RESULTS

In this section, we present some results pertaining to the comparative performance of the proposed formulas in terms of the principal thermal variables used in thermal analysis and design, i.e., the center, surface, and mean temperature distributions, the equivalent "plug-flow" Nusselt number distribution, and the time required to attain steady state conditions. Also, we present an exhaustive analysis of the errors inherent to the approximate solutions and relying on the Grigull and Sandner criteria [8] we determine their corresponding region of validity.

6.1. *Temperature Distribution*

The dimensionless temperature distributions computed by the various methods are compared with the exact solution (the baseline solution) in Figs. 2–5. These two set of figures are similar, the sole exception lies in the range of the abscissas. Figures 2 and 4 may be conceived as a close-up of Figs. 3 and 5, respectively, in the lower left corner. For instance, in Figs. 2 and 4 the abscissa τ extends from 0 to 0.15 in order to capture the details of the sensitive temperature evolution in the early heating stages that are inherent to each method, while Figs. 3 and 5 cover heating periods that extend up to $\tau = 1$.

Inasmuch as the exact full-series solution (FS) is concerned it has been carefully generated by evaluating the infinite series in Eq. (7) with a symbolic mathematic code. To guarantee

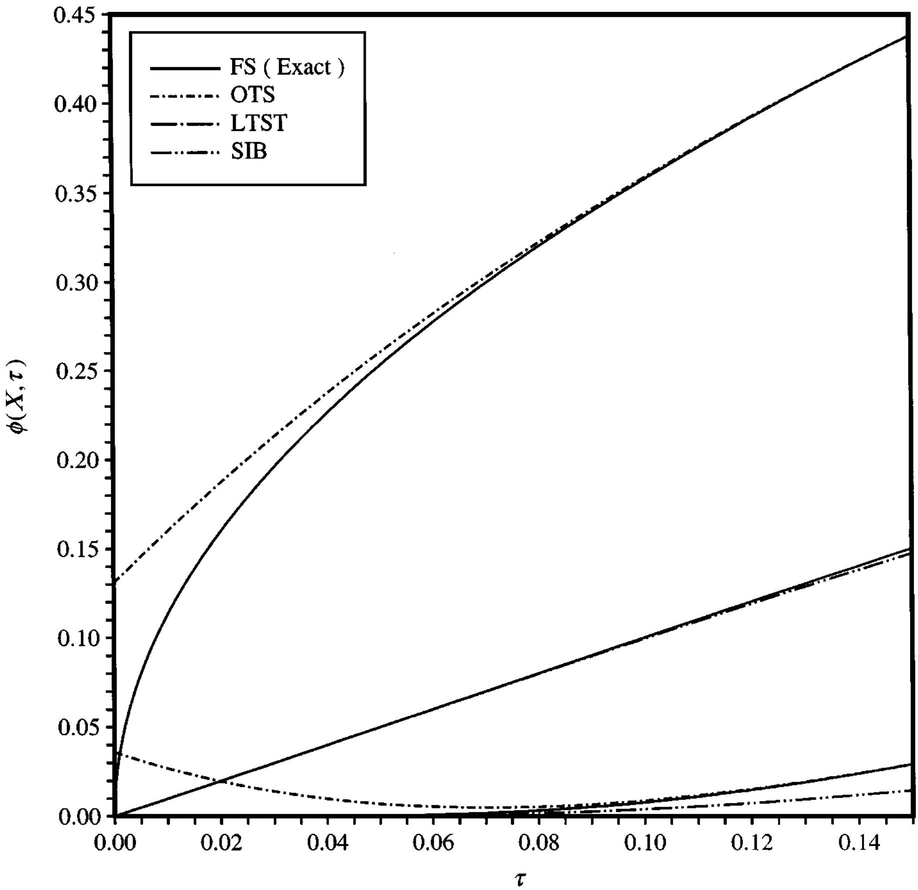

FIG. 2. Center, surface, and mean temperature distributions obtained by using traditional approximate solutions during the early transient period.

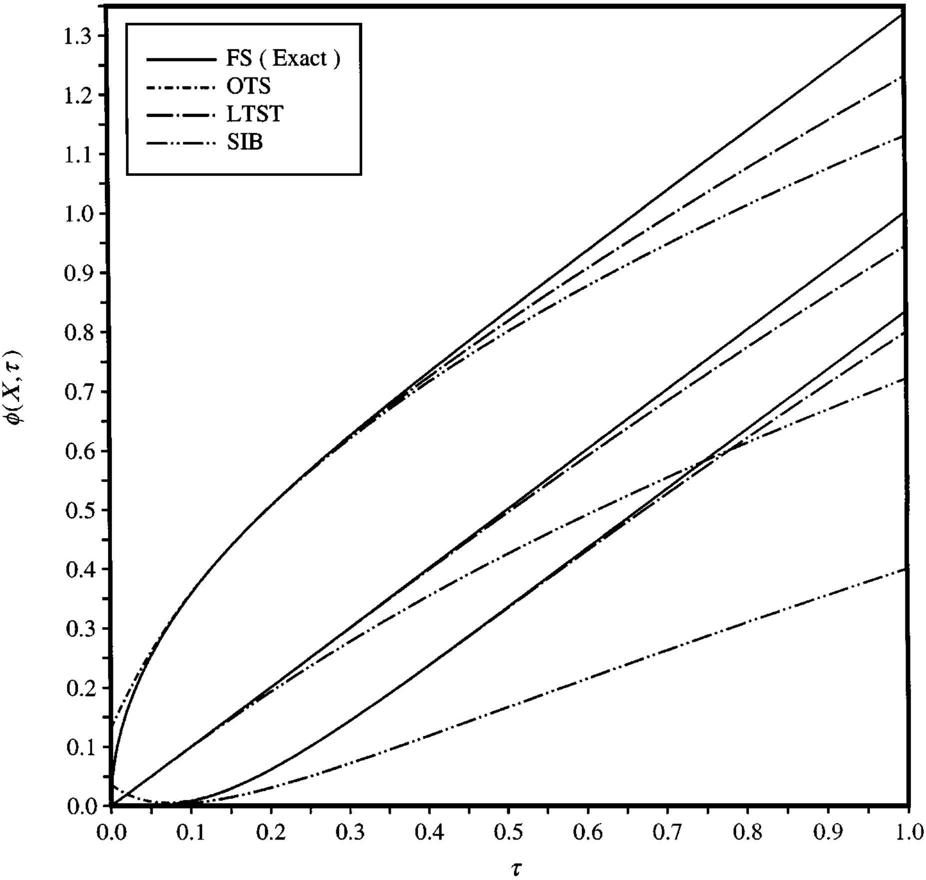

FIG. 3. Center, surface, and mean temperature distributions obtained by using traditional approximate solutions during the entire transient period.

the minimum accuracy of 10^{-7} in the local temperatures at any position and time a maximum of 1300 terms has been retained in the series. For the sake of brevity, two locations inside the slab have been chosen to report the temperatures, namely the center ($X = 0$) and the surface ($X = 1$). At any level of time, the center and surface are respectively associated with the minimum and the maximum temperatures. In addition, the mean temperature distributions with its patented slope of one has been plotted. This global quantity is important because it provides a measure of the bulk thermal energy absorbed by the body within a fixed period of time.

The performance of the traditional approximate solution methods tested, such as OTS, LTST, and SIB, was predominately consistent with expectations from mathematical and physical arenas. The most salient features of each of them have been delineated in the forthcoming paragraphs.

First, we observe in Figs. 2–5 how the OTS (termed the long term solution) exhibits good quality for the center and surface temperatures beyond $\tau = 0.08$. For $\tau < 0.08$, both curves go astray even violating the physics of the problem. The center and surface temperatures at $\tau = 0$ are overpredicted evidencing their inability to retrieve the initial condition. In contrast, the mean temperature overlaps with the exact straight line at all times. This perfect match indicates in a convincing manner that at other locations (different than the center and the

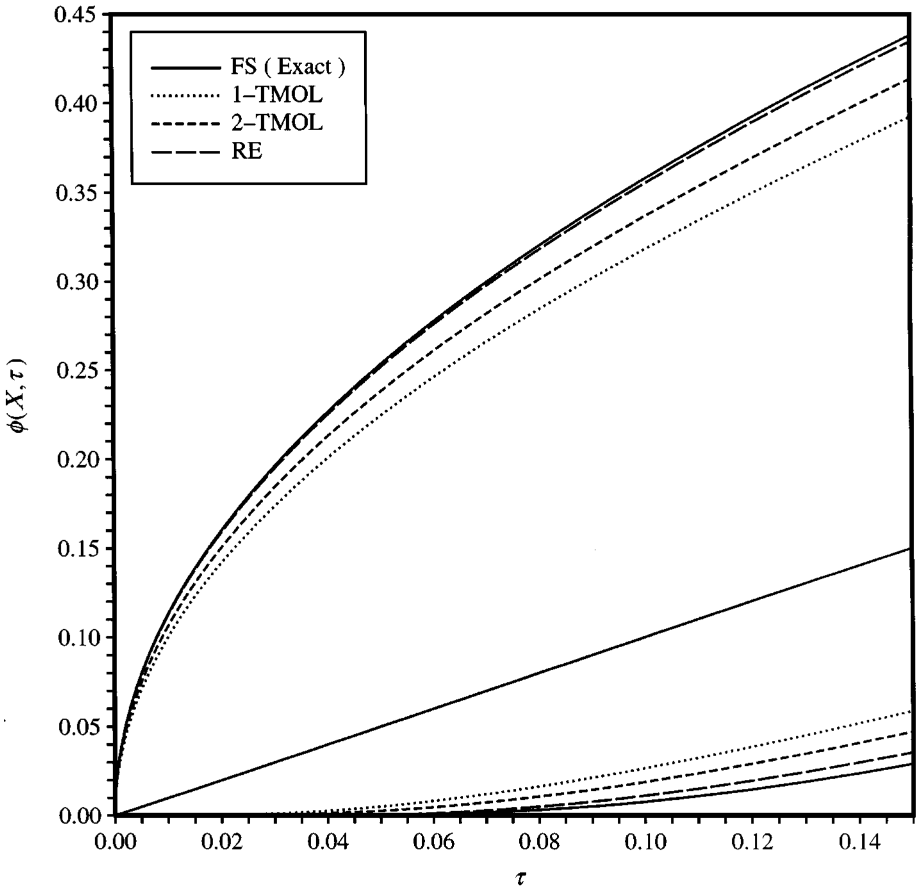

FIG. 4. Center, surface, and mean temperature distributions obtained by using TMOL-type approximate solutions during the early transient period.

surface) the temperatures at early times have to attain negative values for compensatory effects.

Second, the LTST approach gives reasonable estimates for the center and surface temperatures for short times in proximity of 0.7 and 0.4, respectively. These solutions detach from the exact ones for times exceeding these respective values. For larger times, the center temperature deviations grow much slower than their counterparts for the surface temperature deviations. For instance at $\tau = 1$, the former is about half the latter. More or less, the same pattern prevails for the mean temperature and the agreement with the exact straight line is adequate for intermediate values confined to $0 < \tau < 0.5$. Once the upper time is surpassed the approximate curve for the mean temperature starts to deviate slightly.

Third, the center and surface temperatures as predicted by SIB are accurate for short times of 0.07 and 0.3, respectively. The solutions break down dramatically for times exceeding these respective values. As time increases, the center temperature deviations grow more rapidly than their counterparts for the surface temperature deviations. For instance at $\tau = 1$, the former is about twice the latter. These trends are consistent with the physical nature of SIB and are opposed to the ones observed for the LTST. Literally, the pattern displayed for the LTST prevails for the mean temperature here showing an adequate agreement for $\tau < 0.5$.

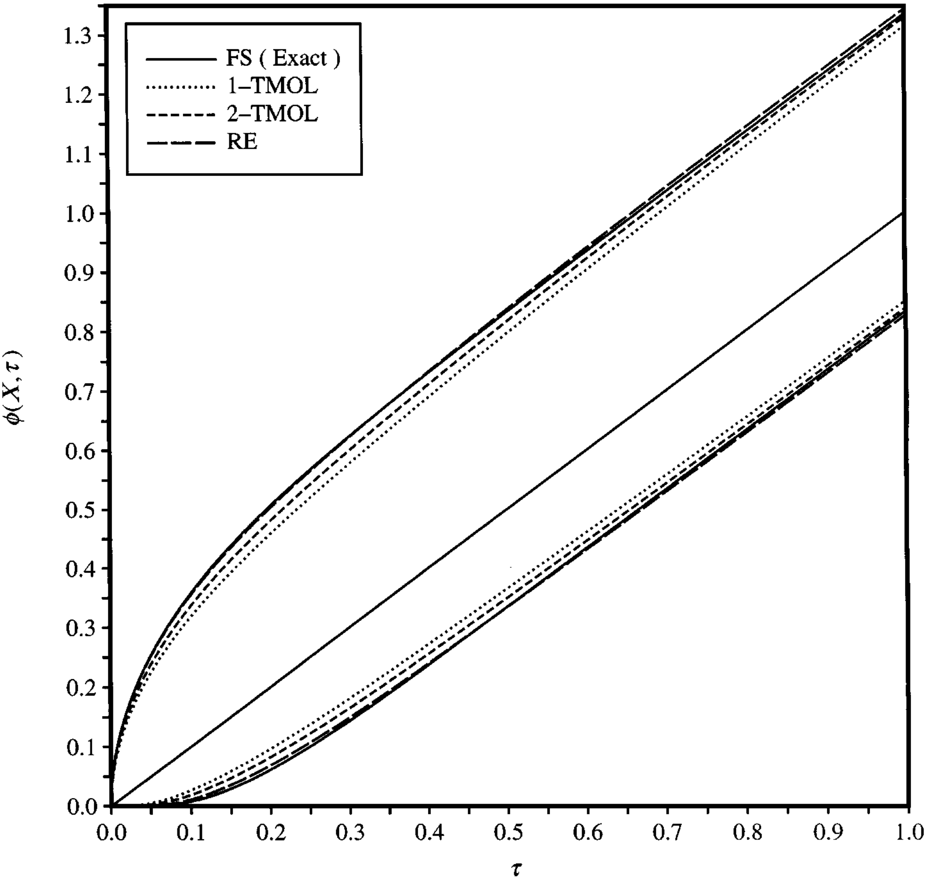

FIG. 5. Center, surface, and mean temperature distributions obtained by using TMOL-type approximate solutions during the entire transient period.

The traits of the contemporary approximate solution methods, namely 1-TMOL, 2-TMOL, and RE were somewhat surprising. Conceptually, the outcome of these solutions are expected to give accurate answers for short times in conformity with the first-order errors inherent to the finite-difference formulation in time. Figures 4 and 5 display the center and surface temperatures produced by 1-TMOL which are always bounded by the exact temperatures curves, i.e., the approximate surface temperature lying below the exact surface temperature as opposed to the approximate center temperature which remains above the exact center temperature. At both locations the deviations tend to grow mildly with time, resembling two thin curvilinear wedges. The temperature calculations for the center and the surface produced by a refined 2-TMOL are remarkably more accurate than those associated with a crude 1-TMOL. Both TMOL procedures share errors of first order. An inspection of the corresponding curves attests that the magnitude of the deviations are cut in half irrespective of the time level.

It is striking that the mean temperatures calculated by 1-TMOL, 2-TMOL, and RE (combining 1-TMOL and 2-TMOL) supply straight lines that perfectly coincide with the exact straight line that emerges from evaluating 1300 terms in the infinite series, the so-called FS.

The temperature results generated by the more refined procedure, the RE, will be discussed next. The surface temperature predictions for the slab computed with the RE compare well with the FS results in its entirety. For this configuration, the center temperatures experienced diminutive discrepancies. The mean temperature distributions computed with the RE collapse to the exact temperature distributions for all times.

Considering the first-order time-accuracy of the 1-TMOL and 2-TMOL expressions in conjunction with the second-order time-accuracy of the RE relations, it should be anticipated that the behavior should be accurate for short times only. Surprisingly, the matching between the RE and the FS solutions is perfect in the entire time domain. Finally, as can be corroborated in Lemma A.1, all TMOL-type solutions yield the exact value for the mean temperature.

6.2. *Equivalent Nusselt Number Distribution*

We recognize the great similarity between unsteady heat conduction in solid bodies and steady duct-flow heat convection. It can be shown that the mathematical formulation of the unsteady heat conduction is identical to that of the thermally developing duct flow with an uniform velocity profile (plug-flow). Hence, the heating period in unsteady heat conduction is analogous to steady heat convection in the entrance region of a duct. Because of this similarity, studying the equivalent Nusselt number history obtained in unsteady heat conduction problems gives the Nusselt number distributions for the equivalent heat convection process in a duct.

Accordingly, the definition for the equivalent local Nusselt number [9] is

$$\mathrm{Nu}_{eq}(\tau) = \frac{1}{\phi(1,\tau) - \bar{\phi}(\tau)} \tag{24}$$

and its asymptotic equivalent value, Nu_∞, corresponds to

$$\mathrm{Nu}_\infty = \lim_{\tau\to\infty} \mathrm{Nu}_{eq}(\tau). \tag{25}$$

In particular for a slab configuration, $\mathrm{Nu}_\infty = 3$.

The equivalent Nusselt number distributions are presented in Figs. 6 and 7. In order to facilitate the comparisons, the distributions have been separated in two parts. Figure 6 compares the equivalent Nusselt number obtained by using the traditional approximate solutions with the one given by the baseline exact solution (FS). Figure 7 shows the comparison of the TMOL-type solutions and the FS solution.

Again the intrinsic tendencies of the traditional approximate solutions, OTS, LTST, and SIB, are evidenced in Fig. 6. Here, it can be seen that the OTS curve (the long time solution) coincides with the FS curve for large times ($\tau \geq 0.1$) and as time decreases the OTS curve stays way below the FS curve. In contrast, the curves associated with the LTST and the SIB solutions overlap the FS curve for short and intermediate times ($\tau < 0.1$). Thereafter, the LTST curve moves up sharply while the SIB curve moves down mildly in an almost symmetrical fashion. These two asymptotic behaviors are counterproductive. As a side comment, it should be added that the LTST solution is more accurate than the SIB solution at the expense of dealing with a more complicated mathematical procedure which is coupled with a more elaborate evaluation of the pertinent expressions.

As noted with the preceding discussion involving the TMOL-type relations for the local and mean temperatures, the same patterns prevail for the estimation of the local equivalent Nusselt numbers. Figure 7 indicates that the crudest 1-TMOL curve is the least accurate, followed by the 2-TMOL curve. In both, maximum deviations occur in the vicinity of

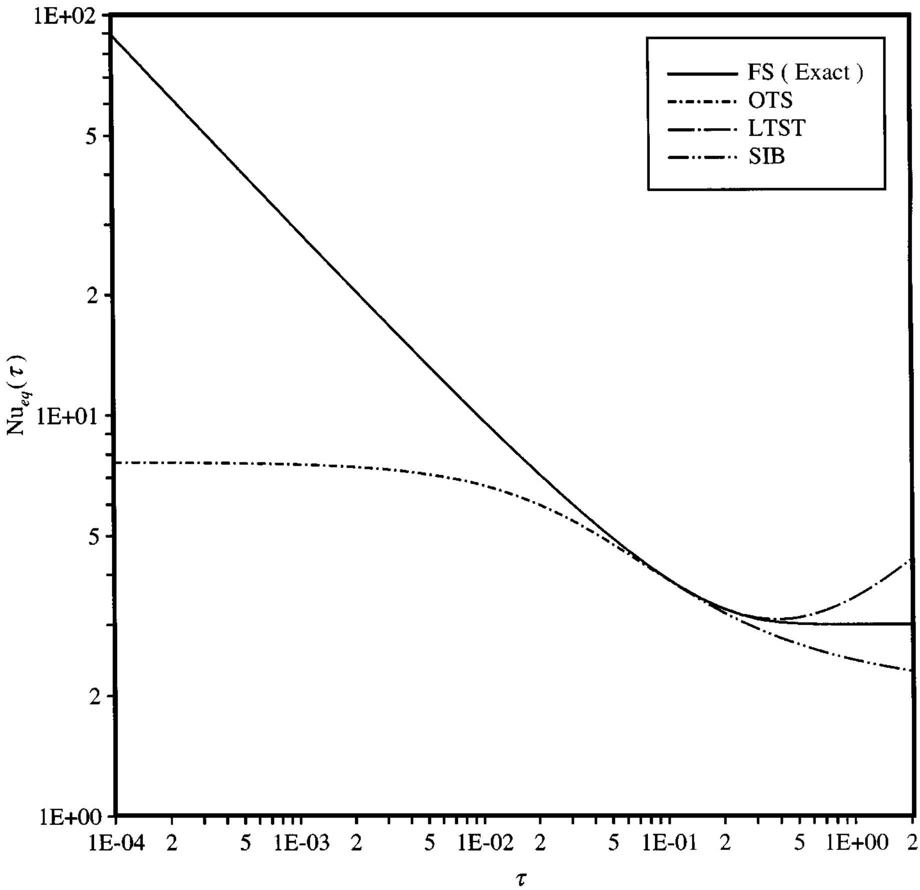

FIG. 6. Comparison of the equivalent Nusselt number distributions obtained by using the traditional approximate solutions.

$\tau = 0.2$. At other times the deviations are uniform. The RE curve gives the best prediction and the deviations from the FS curve are imperceptible to the scale of the graph. Literally these well-behaved trends of the three TMOL schemes contrast markedly with the trends corresponding to the OTS, LTST, and SIB solutions.

6.3. *Time to Reach Quasi-Steady-State Condition*

In this section, we compare the value of the dimensionless time required to reach a quasi-steady-state condition computed by using the approximate solutions with the exact solutions calculated by evaluating the full-series (FS). Notice that for most of the approximate solutions studied here (which are formally valid for small values of time), the prediction of the ending of the transient period could be considered out of order. However, we included these results because they could be an indication of how well the approximate formulas behave for moderate-to-long times.

Following the same analogy between unsteady heat conduction and internal duct flow discussed previously, we define τ_s as the dimensionless time required to achieve a value of the local equivalent Nusselt, Nu_{eq}, equal to 1.05 of the asymptotic equivalent Nusselt, $\mathrm{Nu}_\infty = 3$.

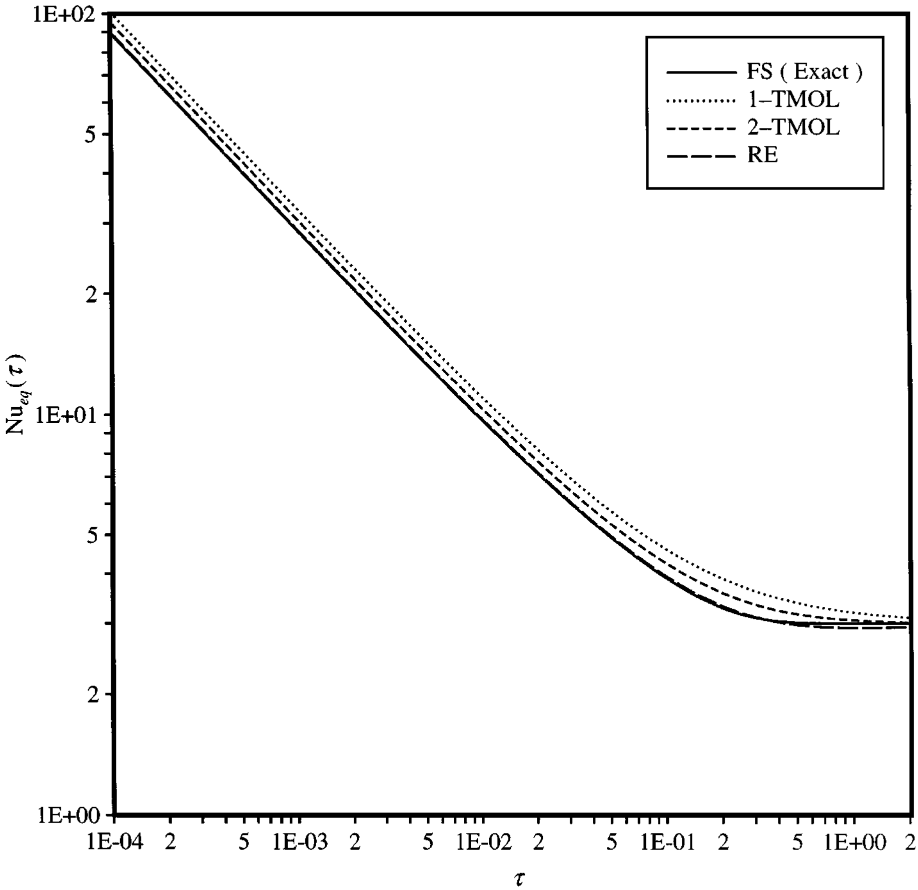

FIG. 7. Comparison of the equivalent Nusselt number distributions obtained by using the TMOL-type approximate solutions.

The computed values of τ_s are summarized in Table 1. As expected, the OTS formula predicts an excellent estimate of τ_s whereas LTST solution underestimates τ_s. The opposite response is found by using the SIB expression; it slightly overestimates τ_s. Prediction of the time to attain fully established conditions by using 1-TMOL and 2-TMOL is remarkably

TABLE 1
Prediction of the Quasi-steady-state Conditions

	Slab	
Method	τ_s	%
FS (Exact)	0.25806	—
OTS	0.25805	−0.004
LTST	0.21512	−16.641
SIB	0.26499	2.685
1-TMOL	1.30510	405.735
2-TMOL	0.53143	105.933
RE	0.27243	5.568

overestimated. On the contrary, RE solution slightly overestimates the value of τ_s with an error always less than 5.6%.

6.4. *Analysis of Error and Range of Validity*

In order to put these unexpected responses of the TMOL-type solutions in perspective, it is important to carry out a systematic study of the errors associated with the various solutions. This study will lead to the subsequent definition of the regions of validity for the different approximate formulas presented.

To the authors knowledge, the range of validity has been established only for the OTS solution. In fact, Grigull and Sandner [8], tolerating an absolute error of about 0.01 in $\phi(0, \tau)$ and $\phi(1, \tau)$, have pointed out that the one-term-of-series (OTS) solution of the unsteady heat conduction equation with Robin boundary conditions is rigorously valid for $\tau > 0.24$ in the case of a slab.

To commence the analysis, we define the local error in the solution as the difference between the approximate solution and the exact (FS) solution, i.e.,

$$\varepsilon(X, \tau) = \Phi(X, \tau) - \phi(X, \tau), \tag{26}$$

where $\Phi(X, \tau)$ represents the approximate solution and $\phi(X, \tau)$ corresponds to the exact solution. In the present study, computing the percentages has been avoided because of the very small numbers that intervene in the calculations.

The analysis of errors is performed in terms of the maximum norm (infinity norm) of the error in the solution at a given time τ. This is defined by

$$\|\varepsilon(\cdot, \tau)\|_{\max} = \sup_{0 \le X \le 1} |\varepsilon(X, \tau)|.$$

Another illustrative quantity, the maximum value of the infinite norm of the error, defined by

$$\varepsilon_{\max} = \sup_{\tau \ge 0} \|\varepsilon(\cdot, \tau)\|_{\max},$$

is also contemplated for comparative purposes. Notice that $\varepsilon_{\max}$ cannot be computed unless $\|\varepsilon(\cdot, \tau)\|_{\max}$ remains bounded for $\tau \ge 0$.

A graphical representation of the values of the maximum norm of the error in the solutions produced by the different approximate formulas is presented in Fig. 8.

The OTS solution displays its typical behavior, i.e., $\|\varepsilon(\cdot, \tau)\|_{\max}$ tends to a maximum for short times. Specifically, for $\tau > 0.05$ the norm of the error for this approximate solution decreases rapidly.

The errors for all the other approximate solutions initiate at zero for $\tau = 0$ and increase gradually. As expected, those errors associated with the LTST and SIB solutions increase substantially as τ grows. The SIB solution is fairly accurate for small and moderate values of time.

Turning the attention to the three TMOL-type solutions it may be seen that they share a similar qualitative behavior. The norm of the error increases consistently to a maximum and then declines rapidly to zero. For short times, the error of 2-TMOL is about half of the error of 1-TMOL while the error of RE is one fourth of the error of 1-TMOL approximately.

Invoking the criteria of Grigull and Sandner [8], i.e., stipulating a maximum norm of the error of 0.01, the region of validity of the approximate solutions can be easily computed.

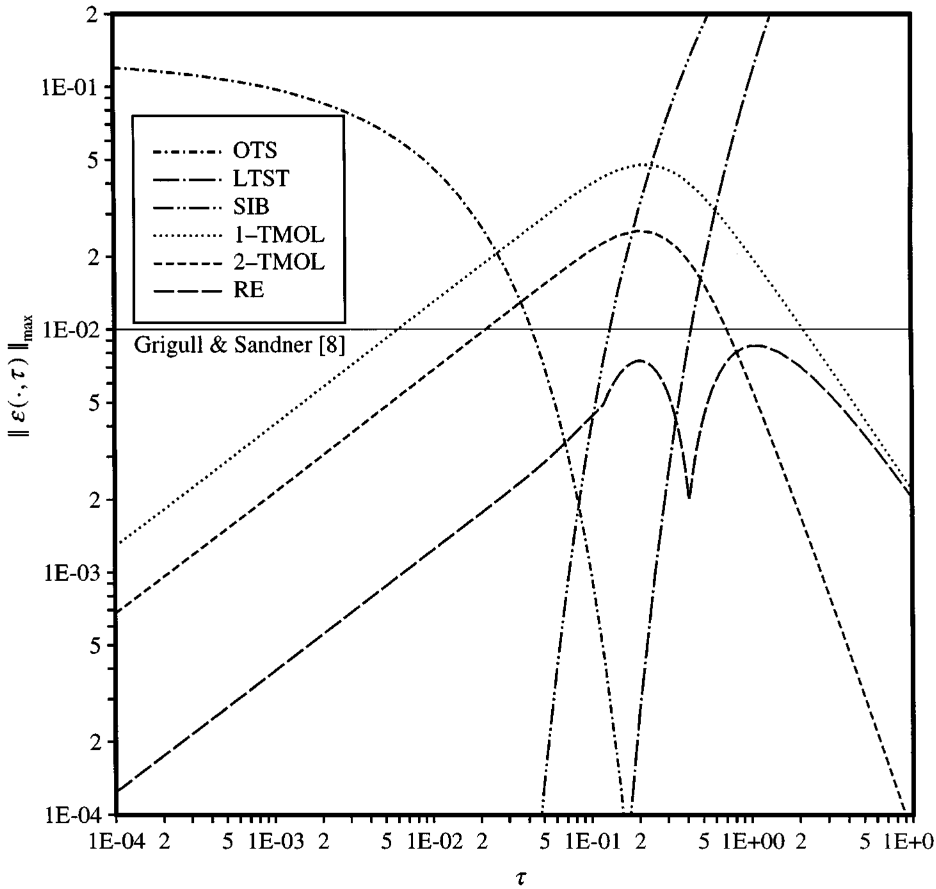

FIG. 8. Maximum norm of the error in the solution.

Our results are summarized in Table 2. We present also the maximum values of the infinite norm of the error, ε_{max}, over the heating period and the dimensionless values of time at which they are attained.

In sum, it was found that for the unsteady heat conduction with Neumann boundary condition (uniform heat flux at the surface) the region of validity of the OTS, the LTST, and the SIB solutions are formally $\tau > 0.04249$, $\tau \leq 0.42699$, and $\tau \leq 0.13172$, respectively.

TABLE 2
Grigull and Sandner Criteria for the Slab

Slab		
Method	Range of validity	ε_{max}
OTS	$\tau > 0.04249$	0.13069 at $\tau = 0.00000$
LTST	$\tau \leq 0.42699$	unbounded
SIB	$\tau \leq 0.13172$	unbounded
1-TMOL	$\tau \leq 0.00607$	0.04779 at $\tau = 0.22050$
2-TMOL	$\tau \leq 0.02181$	0.02554 at $\tau = 0.20715$
RE	$\tau \geq 0$	0.00858 at $\tau = 1.08230$

On the other hand, 1-TMOL is formally acceptable for $\tau \leq 0.00607$. The correctness of the 2-TMOL formula improves remarkably with respect to the companion range of 1-TMOL. Ordinarily, its adequacy is not sufficient to supplement the OTS solution for moderate values of time meaning that 2-TMOL is perfectly valid for $\tau \leq 0.02181$. According to Grigull and Sandner criteria [8], RE solution is adequate over the entire heating period. From these results it is obvious that the RE formula, Eqs. (22), is appropriate to supplement the simplistic OTS solutions for small to moderate values of time.

The maximum error, $\varepsilon_{\max}$, for the OTS solution reaches 0.13069 at $\tau = 0$. In contrast, no maximum error is ever attained for the LTST and the SIB solutions, simply because the error continues to grow indefinitely with τ.

The 1-TMOL procedure yields a maximum error of 0.04779 at $\tau = 0.22050$. Similarly, the 2-TMOL approach reaches a maximum error of 0.02554 at $\tau = 0.20715$. As pointed out before, this maximum error attached to the 2-TMOL formula is roughly half of the maximum error for the 1-TMOL formula. The infinite norm of the error for the RE reaches a maximum of 0.00858 at $\tau = 1.08230$.

Relying on the criteria of Grigull and Sandner [8], the LTST, the SIB, and the RE formulas are acceptable ways to supplement the OTS solution.

Ultimately, to complete our error analysis we present some results in terms of the 1-norm of the error in the solution. Accordingly, we defined it as

$$\|\varepsilon(\cdot, \tau)\|_1 = \int_0^1 |\varepsilon(\xi, \tau)|\, d\xi.$$

This norm can be envisioned as a measure of the deviations in the local thermal energy stored as computed by the different methods. Notice that if we drop the absolute value, this formula reduces to the difference between the approximate and exact mean temperature distributions, $\overline{\Phi}(\tau) - \overline{\phi}(\tau)$.

Attention is now turned to Fig. 9 where results in terms of the 1-norm are plotted. Because all norms are equivalent, the behavior is similar to those of Fig. 8. At this juncture, it should be stressed that the importance of this figure lies in its ability to estimate the accuracy of the different methods to predict the local thermal energy stored. Moreover, they also encompass a more global measure of the performance of the different methods.

In addition, the two main characteristics of the TMOL-type methods are also shown here. For this type of approach the norm of the error in the solution is bounded and, even more important, for large values of time exhibits attractive declining patterns.

The reader should bear in mind that if a specific engineering application can stand the maximum error predicted by the norms, the tedious evaluation of the series can be completely replaced by a single evaluation of any of the TMOL-type formulas.

7. CONCLUSIONS

At the beginning, the principal objective of this comparative study was to explore a quick, direct, and reliable computational procedure for the determination of the temperature-time history for short periods of time within slabs heated by an applied heat flux at their surfaces. The accuracy of the systematic calculations based on a Richardson extrapolation of the first two TMOL solutions exceeded our expectations and convincingly demonstrated that these algebraic solutions are not only adequate for short times, but they extend to the entire

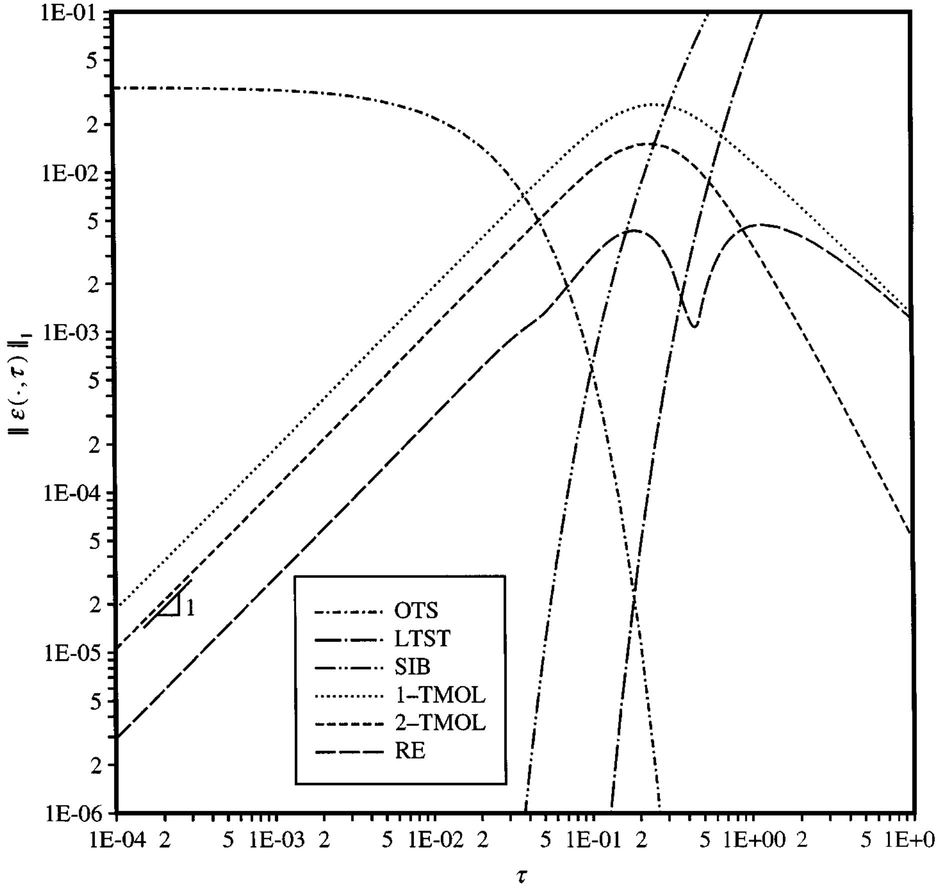

FIG. 9. 1-norm of the error in the solution.

time domain without imposing restrictions. TMOL left no doubt about its credentials as a top contender method for solving the diffusion equation subjected to a uniform heat flux boundary condition.

The simplicity exhibited by the implementation of TMOL along with its algebraic solutions has been proven to be attractive to instructors of graduate courses on conduction heat transfer. These attributes contrast with the full series solution, FS, and other equally elaborate solution methods, like for instance the Laplace Transform solution, LTST, and the semi-infinite body solution, SIB.

Although the temperature distributions for the infinite cylinder and the sphere have not been presented explicitly here, it may be inferred that the tendency of the errors should be similar to that found in the present case.

A. APPENDIX

Consider the following lemma,

LEMMA A.1. *For a slab heated with a uniform heat flux (Neumann boundary condition), the solutions resulting from applying TMOL-type methods to the time-dependent,* *one-dimensional heat conduction equation always yield* $\bar{\Phi}(\tau) = \bar{\phi}(\tau)$. *Moreover, for all* τ, *the integral of the time-truncation error over the spatial domain is zero.*

Proof. Consider a generalized version of Eq. (1)

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial X^2} + \frac{c}{X}\frac{\partial \phi}{\partial X} \qquad \text{in } 0 \leq X \leq 1, \tau > 0,$$

where c is a geometric parameter, e.g., $c = 0$, 1, and 2, for slab, cylinder, and sphere. Integrate both sides of the equation over the entire volume, e.g.,

$$(1+c)\int_0^1 \frac{\partial \phi}{\partial \tau}\xi^c \, d\xi = (1+c)\int_0^1 \left(\frac{\partial^2 \phi}{\partial \xi^2} + \frac{c}{\xi}\frac{\partial \phi}{\partial \xi}\right)\xi^c \, d\xi. \tag{27}$$

Applying the Leibnitz theorem to both sides of the equation and introducing the boundary conditions, Eqs. (3)–(4), we obtain

$$\frac{d\bar{\phi}}{d\tau} = (1+c)X^c \frac{\partial \phi}{\partial X}\bigg|_0^1 = (1+c). \tag{28}$$

So far, this relationship is exact.

When we use TMOL-type semi-analytical solutions we have to substitute the LHS of Eq. (1) by a two-point backward finite difference approximation, e.g.,

$$\frac{\partial \phi}{\partial \tau} = \frac{\phi(X,\tau) - \phi(X, \tau - \Delta\tau)}{\Delta\tau} + \vartheta(X, \tau, \Delta\tau), \tag{29}$$

where, ϑ is the time-truncation error. Inserting the above formula into Eq. (27), neglecting the truncation error, and performing the integration results in

$$\frac{\bar{\Phi}(\tau) - \bar{\Phi}(\tau - \Delta\tau)}{\Delta\tau} = (1+c)X^c \frac{\partial \Phi}{\partial X}\bigg|_0^1 = (1+c). \tag{30}$$

To arrive at this relation we have employed Eqs. (3)–(4) again. Notice that besides the restriction that $\tau \geq \Delta\tau$, τ and $\Delta\tau$ are arbitrary quantities. Then, without loss of generality we can assume that $\Delta\tau = \tau$ in Eq. (30). Next, using the initial condition, Eq. (2), the above equation reduces to the exact value, $\bar{\Phi}(\tau) = (1+c)\tau = \bar{\phi}(\tau)$.

Finally, comparing Eqs. (28) and (30), after introducing Eq. (29), leads to

$$\bar{\vartheta}(\tau, \Delta\tau) = (1+c)\int_0^1 \vartheta(\xi, \tau, \Delta\tau)\xi^c \, d\xi = 0. \qquad \blacksquare$$

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